The Second Fundamental Theorem of Calculus

-In all the definite integral examples we've done so far we've had the bounds be defined as constants from *a* to *b*.

$$\int_a^b f(x)\,dx$$

-In a different situation there might also be a need to make the upper limit of integration be variable as opposed to a finite number.

$$F(x) = \int_a^x f(t)\,dt$$

-Take note how "little f" is a function of *t* but the resulting "big F" is a function of the upper limit of integration, *x*.

Example

Evaluate $\int_0^x \cos(t)\,dt$ at $x = 0, \pi/6, \pi/4, \pi/3, \pi/2$.

-One way to do this would be to do five separate integrals one for such of the upper limits.

-However, if we treat *x* as a constant to start with we obtain a much simpler solution:

$$\int_0^x \cos(t)\,dt = \sin(t)\Big]_0^x = \sin(x) - \sin(0) = \sin(x)$$

-Now using $F(x) = \sin(x)$ you can obtain:

$$F(0) = 0, F\left(\frac{\pi}{6}\right) = \frac{1}{2}, F\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, F\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, F\left(\frac{\pi}{2}\right) = 1$$

-You can think of this action of evaluation the area under $f(t)=\cos(t)$ from $t=0$ to $t=x$ as accumulating the area under the cosine curve from $[0,\pi/2]$.

-The interpretation of the integral as an **accumulation function** is often used in applications of integration.

-As with other integration we can check the result of our work by differentiating the result.

$$\frac{d}{dx}\sin(x)=\cos=(x) \qquad \int_0^x \cos(t)\,dt$$

-Note that this is the same thing as our original problem with the variable now a x instead of a t.

-This leads to a **Second Fundamental Theorem of Calculus**.

-If f is continuous on the open interval I containing a, then for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$$

<u>Example</u>

Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\ dt\right]$

-Since $\sqrt{t^2+1}$ is continuous we can use the Second Fundamental Theorem of Calculus and write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\ dt\right]=\sqrt{x^2+1}$$

Example

Find the derivative of $F(x)=\int_{\pi/2}^{x^3}\cos(t)\,dt$.

$$\int_{\pi/2}^{x^3}\cos(t)\,dt=\sin(t)\Big]_{\pi/2}^{x^3}$$

$$=\sin\left(x^3\right)-\sin\left(\frac{\pi}{2}\right)$$

$$=\sin\left(x^3\right)-1$$

-Now differentiating this we get

$$\frac{d}{dx}\left[\sin\left(x^3\right)-1\right]$$

$$=\cos\left(x^3\right)\left(3x^2\right)$$

Using this result we can generalize the 2nd Fundamental Theorem to be:

$$\frac{d}{dx}\left[\int_a^u f(t)\,dt\right]=f(u)\frac{du}{dx}$$

Example

Find the derivative of $F(x)=\int_{x}^{x^2}\cos(t)\,dt$.

To evaluate when the bounds of integration are both functions, use some integral properties to rewrite.

$$\int_{x}^{x^2} \cos(t)\, dt = \int_{x}^{0} \cos(t)\, dt + \int_{0}^{x^2} \cos(t)\, dt$$

$$= -\int_{0}^{x} \cos(t)\, dt + \int_{0}^{x^2} \cos(t)\, dt$$

$$= -\cos(x) + 2x\cos(x^2) + C$$

Average Value of a Function

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

Example

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

$$\frac{1}{b-a}\int_{a}^{b} f(x)\,dx = \frac{1}{4-1}\int_{1}^{4} (3x^2 - 2x)\,dx$$

$$= \frac{1}{3}\left[x^3 - x^2\right]_{1}^{4}$$

$$= \frac{1}{3}\left[64 - 16 - (1 - 1)\right] = \frac{48}{3} = 16$$